THE MAGNIFICENT

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INTRODUCTION

It's as easy as pi, $\pi = \frac{\phi^2(1+m^2)}{bm}$ = *a composite number*.

$\phi^2 = \frac{b^2}{1+m^2} = 2.618033988749894... = a\ composite\ number.$

$\pi = \frac{b}{m} = 3.141592653589793... = a\ composite\ number.$

$\frac{bm}{1+m^2}$ = A perfect square! $= \frac{integer}{integer} = \frac{\phi^2}{\pi}$

$$\frac{\phi m}{\sqrt{1+m^2}} = \frac{\phi m\cdot\sqrt{1+m^2}}{\sqrt{1+m^2}\cdot\sqrt{1+m^2}} = \frac{bm}{1+m^2} = \frac{bm\cdot\frac{b}{m(1+m^2)}}{(1+m^2)\cdot\frac{b}{m(1+m^2)}} = \frac{\frac{b^2m}{m(1+m^2)}}{\frac{b(1+m^2)}{m(1+m^2)}} = \frac{\frac{b^2}{1+m^2}}{\frac{b}{m}} = \frac{\phi^2}{\pi}.$$

$$\frac{\phi m}{\sqrt{1+m^2}} = \frac{0.972208125181406...}{1.166631876039400...} = \frac{bm}{1+m^2} = \frac{1.134208988981134...}{1.361029934191217...} = \frac{\phi^2}{\pi} = \frac{2.618033988749894...}{3.141592653589793...}.$$

The composite number $\phi^2\cdot\left(\frac{1+m^2}{bm}\right)$, when substituted for π, is shown to satisfy and clarify Euler's famous equation $e^{\pi i} = -1$. Since things equal to the same thing are equal to each other then $\pi = \phi^2\cdot\left(\frac{1+m^2}{bm}\right)$.

The composite nature of the numbers ϕ , π, m and b, as described above, is shown by the approximations: $Slope\ m = 0.600857665500921...$, b the $y - Intercept = 1.887650027790808...$, $\phi = 1.61803398874989484...$ and $\pi = 3.14159265358979323...$.

TABLE OF CONTENTS

- **EULER'S PROOF IS NOT ONLY CONFIRMED, IT IS CLARIFIED**.
- **TRUTH EXISTS WHETHER WE FIND IT OR NOT**.

A mathematical paper by Roger Logan, 141 Washington Drive, Warminster, PA 18974. r-logan@ispwest.com.

| Introduction and Table of Contents | | Part 1 - The Narrative | | Part 2 - The Math | | Part 3 - The Proof | | Glossary | | Footnotes |

Top

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PART 1 (THE NARRATIVE)

Two of the world's most famous irrational numbers — **pi**, represented by the Greek letter π (pronounced pie), and **phi**, represented by the Greek letter ϕ (pronounced fi) — are found to be mathematically related to each other by a perfect square in such a way that when π is multiplied by this perfect square the product is ϕ^2, and when ϕ^2 is divided by this perfect square the quotient is π.

It's difficult to understand just why this relationship has remained hidden since the dawn of mathematics. [1] It is even more astonishing in view of the fact that the relationship is easily expressed by right triangles using the Pythagorean theorem, the laws governing mean proportionals, and high school – level mathematics (calculus is not required).

π is used by mathematicians as the symbol for the exact numerical ratio of the circumference of a circle to its diameter; ϕ is used as the symbol for a special number, $\frac{1+\sqrt{5}}{2}$. ϕ is called the *golden ratio* or *golden proportion* and is the only number whose reciprocal, and square, are obtained by, respectively, deleting 1 from itself, and adding one (1) to itself:

$$\phi = 1.618033988749894...$$

$$\phi - 1 = 0.618033988749894\ldots = \frac{1}{\phi}$$

$$\phi + 1 = 2.618033988749894\ldots = \phi^2$$

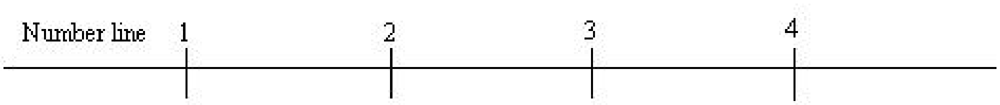
Perfect squares are numbers such as 1, 4, 9, 16, 25, 36,...; but be warned, by substituting the successive numbers 1, 2, 3, 4, 5, 6, ... for the value of n in the expression $991n^2 + 1$ we will never obtain a number that is a perfect square even after spending months or years at what seems like an infinite task. If, on the other hand, we conclude by analogy[2] that all numbers of the form $991n^2 + 1$ are not perfect squares, we are dead wrong, for squares are included, the smallest one having the value obtained when n = 12,055,735,790,331,359,447,442,538,767. Remember, we still need to square this number, multiply the result by 991, and then add 1.

This irrational and transcendental number π appears in many forms other than as the ratio of a circle's circumference to its diameter. π is sometimes expected, as in formulas for other geometric shapes like the ellipse, the cyclonidal arch, and the witch, and sometimes unexpected, as in the determination of the probability of an occurrence.

We use words such as **integer** or **fraction** and adjectives such as **real**, **rational, irrational**, and **transcendental** to describe or modify the word "number," and since we need to introduce other number names, such as **algebraic**, **complex**, and **imaginary**, a short discussion of the number system will help readers to understand those names printed in **bold** above that represent a substantial portion of the number names entered into the number system over the centuries. It appears by hindsight that some of the names, so dubbed, were an unfortunate choice from the descriptive words available, e.g., the name irrational for a non-rational number, and the name imaginary for one not classed as real. (See also the Glossary of Terms).

Over the centuries, mathematicians have expanded the number system four times. Each new expansion was required because the then existing number system was not sufficient to solve certain problems.

The natural number system was not the first number system,[3] but it is a good place to start since it is the class of all positive whole numbers for counting, such as 1, 2, 3, 4, etc. We can represent these natural numbers on a number line using a specific unit length from the start to the first point, and the same unit length from point to point. See Sketch 1 (Sk. 1).



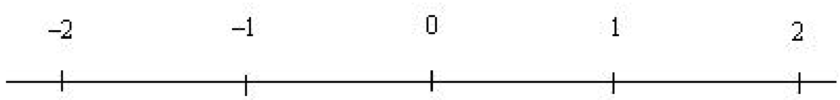
Sk. 1

Class of natural numbers

The natural number system is *closed* under addition and multiplication but is *not closed* under subtraction. This means that all the positive whole numbers, when added to each other or multiplied by each other, have a sum or product that is still contained in the natural number system; but, when they are subtracted from each other then the difference is not included in the system, e.g., subtracting five from three (3 – 5 = – 2) and subtracting four from four (4 – 4 = 0).

1. The class of natural numbers was expanded to the class of integers.

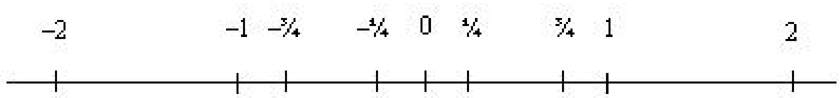
The set of whole numbers that includes the number zero and all the negatives, is 0, – 1, – 2, – 3, – 4, ... By placing this set of whole numbers on the number line with the set of natural numbers we obtain the class of integers (Sk. 2), which is *closed* under addition, multiplication and subtraction, but is *not closed* under division. The class of integers is not closed under the division operation because one integer divided by another integer (both whole numbers) such as 4x = 3 ($x = \frac{3}{4}$) will produce fractions (ratios of whole numbers), which are not in the system. Every algebraic equation of the 1st degree in the form x + b = a can now be solved.



Sk. 2
Class of integers

2. The class of integers was expanded to the class of rational numbers.

By placing the set of fractions on the number line with the class of integers we obtain the class of rational numbers (Sk. 3).

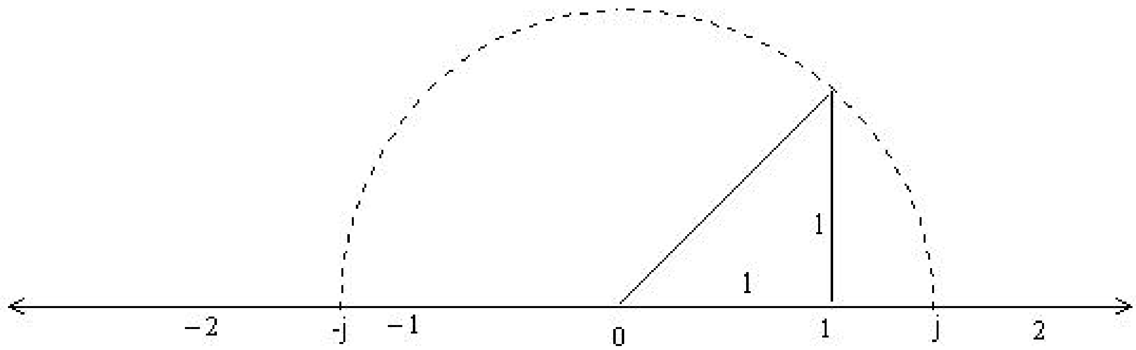


Sk. 3
Class of rational numbers

Every algebraic equation of the 1st degree in the form ax = b can now be solved where $a \neq 0$.

3. The class of rational numbers was expanded to the class of real numbers.

The class of rational numbers could solve equations of the first degree in the form of ax + b = 0, but could not solve (provide points on the number line) for irrational numbers such as $\sqrt{5}$ since such numbers cannot be expressed as the ratio of two integers. This radical sign with the number five inside represents an algebraic equation of the 2nd degree $x^2 - 5 = 0$. The question being asked by this equation is, "What number multiplied by itself equals 5? " There is no fraction (number with integers for the numerator and denominator) that will answer this question. Such a number is called an irrational number. Despite this restraint, we can, if given a unit length, construct on the real number line a segment equal to the length of an irrational number. In the example below (Sk. 4) the points j and – j equal plus and minus the square root of 2 respectively.



SK. 4

Real number line

This real number line has a point for every number and a number for every point. The real number line has *nesting points*[4] for every infinite decimal expansion and for the limit of the sum of every convergent series. The fact that all lengths can be expressed as real numbers is known as the *completeness property of* real *numbers*. Thus the class of real numbers includes the class of rational numbers.

4. The class of real numbers was expanded to the class of the complex number system.

The real number system was able to solve equations of the second degree as noted above, but was unable to solve algebraic equations of the second degree such as $x^2 + 1 = 0$. This equation looks similar to the one noted above, $x^2 - 5 = 0$. However, the question being asked by this equation is: "What number multiplied by itself equals –1? " Let it be sufficient for now to say that the solution is in the form of a + bi (a complex number) where a and b are real numbers and $i^2 = -1$. The complex number system, therefore, includes the class of real numbers.

The age old question is, "Which came first, the chicken or the egg?" This simplistic summary regarding the expansion of the number system makes it appear that the mathematicians' main concern was to expand the number system so as to solve the then current number problems. However, solving the number problems is what caused the creation, discovery or invention of new numbers, thereby forcing the expansion. The author of one of my favorite reference books, when referring to the development of the number system, advises the student to realize that "... mathematics, although called *an exact science*, is arbitrary and man-made — it is a creative art."[5]

This problem solving, leading to the expansion of the number system, had to occur over centuries. For example, once we assume that the natural number system came into existence when man first started to count, and to associate the count with specific objects, it becomes apparent that we are talking about thousands of years before the birth of Jesus Christ.

The expansion to the class of integers required the invention of the number zero. Hindu writing gives evidence that the number zero may have been known before the birth of Christ, but no inscription was found with such a symbol before the 9th century A.D. and zero was not used in decimal form until the 8th – 11th century A.D., nor used in modern notation until 1202 A.D. when *Liber abaci* was published by Leonardo of Pisa also known as Fibonacci.[6] The expansion to the class of integers also required the invention of negative numbers and Fibonacci is credited as being among the first to give practical interpretation to them.

The expansion to the class of rational numbers required the invention of fractions, which were known to the Babylonians and Egyptians as early as 2400 B.C.,[7] but the class of rational numbers also included decimals, which had to wait for the 8th century A.D.

The expansion to the class of real numbers required a solution to the problems of both irrational numbers and transcendental numbers. The irrational number problems were known to Pythagoras about 500 B.C., yet over 2 millennia passed before the complete denuding of their difficulties by the mathematicians Georg Cantor (1845-1918), Richard Dedekind (1831-1916), Karl Weierstrass (1815-1897), and H. C. R. (Charles) Meray 1835-1911).[8] The Greeks knew about transcendental number problems at the time of Euclid of Alexandria (fl. 300 B.C.), yet those problems had to await the work of Ferdinand Lindemann (1852-1939), who proved in 1882 that π is a transcendental number.

The expansion to the class of complex numbers required solution to the problem of imaginary numbers by Italian mathematicians in the 16th century. This solution came before the solutions of the irrational and transcendental problems, which were included in a prior number class. When you consider the mixed-up names and times, it is no wonder that Leopold Kronecker (1823-1891), when referring to negative numbers 2 centuries after they were admitted as respectable members of the mathematical society, and were in general use,[9] remarked,

"Die ganzen Zahlen hat Gott gemacht, alles andere ist Menschenwerk."

("God created the natural numbers, everything else is man's handiwork.")

Need for expansion of the number system has reached its end with regard to algebraic equations. "To be more specific, it is known that every algebraic equation of the form $a_n x^n + a_{n-1} x^{n-1} \cdots a_1 x^1 + a_0 = 0$, whose coefficients are complex numbers, has a solution in the complex number system."[10]

Since π is an irrational number that cannot be expressed as the root of an algebraic equation having rational coefficients, it is called a transcendental number..A transcendental number can only be expressed as an unending decimal, or as the limit of some type of infinite process. A transcendental number cannot satisfy (be a root of) an equation of the form $c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \cdots + c_2 x^2 + c_1 x^1 + c_0 = 0$. This form of equation with rational coefficients can only be satisfied by an algebraic number.[11]

Every real number is classified into either rational or irrational numbers, and then further classified into algebraic or transcendental numbers (see Glossary of terms). All rational numbers are algebraic and all irrational numbers are either algebraic or and transcendental. algebraic irrational numbers are those that satisfy algebraic equations such as $x^2 - 2 = 0$ and $x^3 - 11 = 0$.

All algebraic numbers (including algebraic irrationals) of the first or second degree, or those having a power of two, e.g., x^2, x^4, x^8, x^{16}, x^{32}, x^{64}, x^{128}, ..., if given a line segment of unit length, are numbers that can be constructed by the classical Greek geometrical method of straightedge and compass,[12] that, as postulated by Euclid of Alexandria, about 300 B.C., requires the tools to have the following restrictions:[13]

1. That the straightedge be unmarked, and be used for the purpose of drawing a straight line of indefinite length through any two given distinct points; and

2. That the compass be used to draw a circle with any given point as center, and passing through any given second point.

The most famous problem in the history of mathematics is that of squaring the circle by the use of these Euclidean tools, i.e., constructing a square having an area equal to that of a given circle. This problem captured the attention of mathematicians and amateurs alike from the time of Anaxagoras (ca. 499 – 427 B.C.) until Lindemann proved in 1882 that π is a transcendental number.

As early as the 3rd century B.C., mathematicians discovered this ratio of circumference to diameter for the circle, but they had difficulty putting it into fractional form, because they could not find a fraction, with integers for both numerator and denominator, to express it.

Archimedes of Syracuse (287 B.C. – 212 B.C.), considered one of the three greatest mathematicians, along with Sir Isaac Newton (1642 – 1727) and Carl Friedrich Gauss (1777 – 1855), is credited with the first scientific approach to the computation of π. He used a circle with a unit diameter, and by inscribing and circumscribing polygons (classical method) he calculated that the value of π was between $3\frac{10}{71}$ and $3\frac{1}{7}$, or 3.14, which is correct to two decimal places. The approximate value of the ratio correct to 20 decimal places is the number 3.14159265358979323846....

About the year 150 A.D. Claudius Ptolemy (ca. 85 – ca. 165), Greek author of the *Almagest*, calculated π as 3.1416, and about the year 480 Tsu Ch'ung-chih (430 – 501), a Chinese mathematician, gave the rational approximation of π as $\frac{355}{113} = 3.1415929\ldots$, that is correct to 6 decimal places. In 1579 the French mathematician Francois Viete (1540 – 1603), or, in Latin, Franciscus Vieta, by the classical method using polygons having 393,216 sides, computed π correct to 9 decimal places. He also showed as the limit of an infinite product correct to 10 decimal places.

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

π was calculated to 35 places in 1610 by Ludolph van Ceulen (1540 – 1610) using polygons having 2^{62} sides, and then in 1621 the Dutch physicist Willebrord Snell (1580 – 1626) improved the classical method of computing π by an application of trigonometry, enabling him to match Ceulen's 35 decimal places with polygons of 2^{30} sides.

In the early 17th century, after the co-invention of calculus by Newton and Gottfried Leibniz (1646 – 1716), π cropped up in the purely algebraic devices of convergent, infinite products, and convergent, infinite series, which came into fashion.

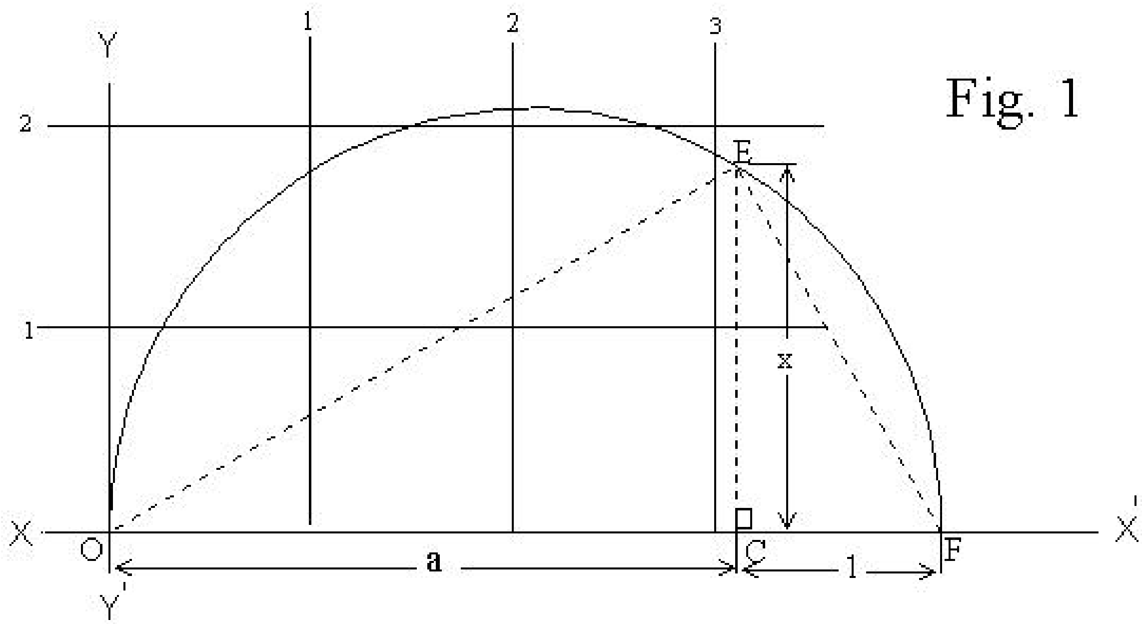
$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$ infinite Product John Wallis (1616 –1703)

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdots$$ infinite Series James Gregory (1638 – 1675)

Modern mathematicians have abandoned the search for a way to put the ratio into fractional form. The attempt to find a rational value for the number π ended in 1767, when the German mathematician Johann Heinrich Lambert (1728 –1777) showed that π is an irrational number.

Even though the search ended to find a rational value for π, the search to "square the circle" by the classical Greek geometrical method of straightedge and compass continued by mathematicians and amateurs alike for another 115 years at which time most mathematicians abandoned their search to square the circle.

The whole point of why mathematicians abandoned the search to square the circle is that π is a transcendental number, i.e., an irrational number that is not, and cannot be, the root of an algebraic equation having rational coefficients. A transcendental number can only be expressed as an unending decimal or as the limit of some type of infinite process; therefore, a line segment equal to a transcendental number cannot be defined. See Figure 1 below for a geometric explanation.



We can obtain the square root of any line segment by adding 1 unit to it, and then constructing a semicircle above the extended line (diameter). A perpendicular, extended to the circle from point C, becomes the mean proportional and divides the triangle into proportional segments such that $\frac{a}{x} = \frac{x}{1}$ or $x = \sqrt{a}$.

This is the prime example of just why mathematicians have stated that the circle cannot be squared. A square of equal size to a unit circle must have a side length equal to $\sqrt{\pi}$, but in Figure 1, for x to equal $\sqrt{\pi}$, then *a* must equal π, a transcendental number that cannot be constructed.

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PART 2 (THE MATHEMATICS)

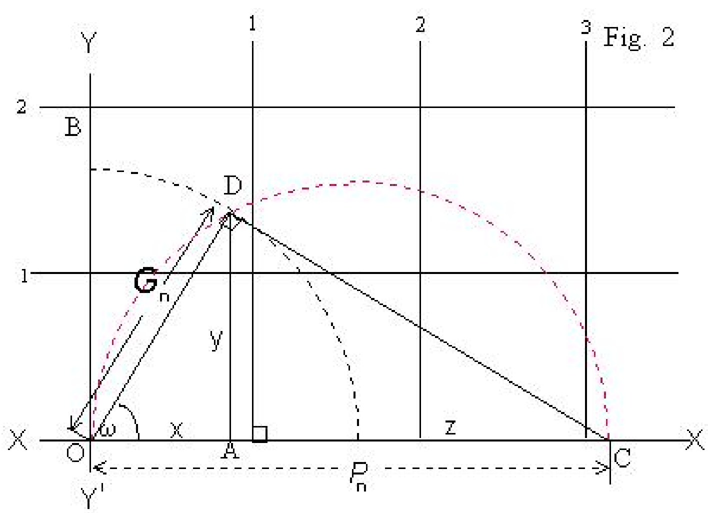
DISCOVERY OF THE PI/PHI STRUCTURE.

- **ELEMENTS OF THE *PI/PHI* STRUCTURE.**
- **LIMITS OF THE PI/PHI STRUCTURES LINE SEGMENTS.**
- **PI AND PHI ARE BOTH COMPOSITE NUMBERS WITH THE SAME IRRATIONAL FACTOR.**

Many mathematicians believe that, because the number π is transcendental and not algebraic, and cannot be expressed in any form other than that of an unending decimal or as the limit of an infinite sequence, it cannot be constructed as a line segment by any method whatsoever. The fact that a structure could exist, providing insight into this matter, is nowhere to be acknowledged.

We are taught in high school geometry class that if the altitude be drawn to the hypotenuse of a right triangle, each leg is the mean proportional between the whole hypotenuse and the adjacent segment. This then is the essence of what I call the π/ϕ structure.

When two infinite sequences, the P sequence $P_1, P_2, P_3, \dots P_n$ and the G sequence $G_1, G_2, G_3, \dots G_n$, having limits of π and ϕ, respectively, are plotted in a system of rectangular coordinates (see Fig. 2 below), a mean proportional relationship, as described above, is shown to exist between the corresponding terms of each sequence. Therefore, the proportion $\frac{\overline{OC}}{\overline{OD}} = \frac{\overline{OD}}{\overline{OA}}$ is established for the first and the successive corresponding terms of these two sequences.



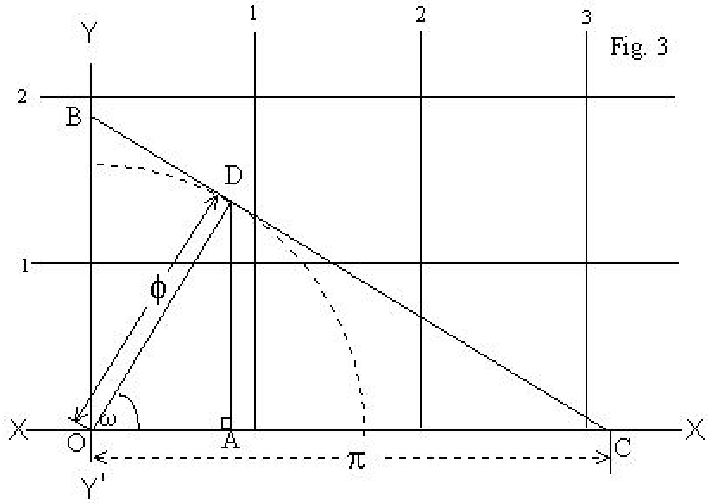
Since the limits of these two sequences are in the same proportion as their terms, then, at the limit $\frac{p_n}{G_n} = \frac{G_n}{X_n}$, or $x_n = \frac{\phi^2}{\pi}$; and ϕ is shown to be the mean proportional between π and line segment OA.

We have now identified (in x and y) the limits of three line segments of the π/ϕ structure:

$$\overline{OC} \to \pi, \ \overline{OD} \to \phi, \text{ and } \overline{OA} \to \frac{\phi^2}{\pi}.$$

By extending line segment CD to intercept the y-axis at B (see Fig 3. below), and by the use of the Pythagorean theorem and the propositions in plane geometry involving mean proportionals, we can now identify the remaining limits (in x and y) for all of the π/ϕ structure's line segments; and then, by use of the slope intercept form ($y = mx + b$), as limits in b and m. This procedure reveals $\overline{OA} = \frac{bm}{1+m^2} = \frac{\phi^2}{\pi}$, allowing us to show that $\frac{bm}{1+m^2}$ is not only a simpler form of $\frac{\phi^2}{\pi}$, but is also a perfect square, a rational fraction having integers for both numerator and denominator.

THE PI / PHI STRUCTURE



We indicate below all the *limits*, of the line segments of the π/ϕ structure (Fig. 3 above) expressed in x and y as proportions of π and ϕ and expressed in b and m as proportions of *b* and *m*. Approximations for ϕ, π, b, and m are

ϕ = 1.61803398874989484... π = 3.14159265358979323... b = 1.88765002779080832... m = .600857665500921494...

$$\overline{OA} = \frac{bm}{1+m^2} = \frac{\phi^2}{\pi}$$

$$\overline{OC} = \frac{b}{m} = \pi$$

$$\overline{AD} = \frac{b}{1+m^2} = \sqrt{\frac{\phi^2\pi^2-\phi^4}{\pi^2}}$$

$$\overline{OD} = \frac{b}{\sqrt{1+m^2}} = \phi$$

$$\overline{AC} = \frac{b}{m(1+m^2)} = \frac{\pi^2-\phi^2}{\pi}$$

$$\overline{BD} = \frac{bm}{\sqrt{1+m^2}} = \frac{\phi^2}{\sqrt{\pi^2-\phi^2}}$$

$$\overline{CD} = \frac{b}{m\sqrt{1+m^2}} = \sqrt{\pi^2-\phi^2}$$

$$\overline{BC} = \frac{b\sqrt{1+m^2}}{m} = \frac{\pi^2}{\sqrt{\pi^2-\phi^2}}$$

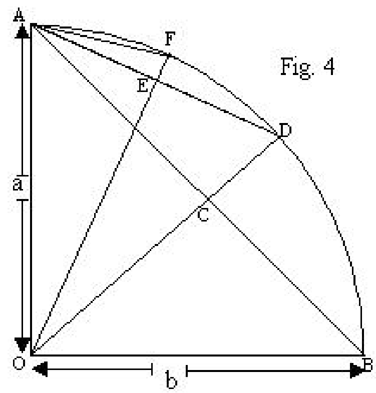
$$\overline{OB} = b = \frac{\pi^2\phi^2}{\pi^2-\phi^2}$$

$$\sin\omega = \frac{b}{\phi(1+m^2)}$$

$$D_n\left(\frac{bm}{1+m^2}, \frac{b}{1+m^2}\right) = D_n\left(\frac{\phi^2}{\pi}, \sqrt{\frac{\phi^2\pi^2-\phi^4}{\pi^2}}\right)$$

$$\cos\omega = \frac{bm}{\phi(1+m^2)}$$

The infinite *P* sequence $P_1, P_2, P_3, \ldots P_n$, that converges on π as its limit, is derived from a computer program designed to calculate the area of a unit circle (Fig. 4 below).



This computer program provides a printout of the areas of an infinite sequence of regular polygons inscribed in a circle (terms of the *P* sequence) as they approach the limit π:

Step	Command
10	$a = 1$
20	$b = 1$
30	l = 4
40	*for* $t = 1$ *to* ∞
50	$h = \sqrt{a^2 + b^2}$
60	$P = P + \frac{a \cdot b \cdot l}{2}$
70	*print* P
80	$a = 1 - \sqrt{1 - (\frac{h}{2})^2}$
90	$b = \frac{h}{2}$
100	$l = 2 \cdot l$
110	*Next* t
120	*end*

Base calculations (see Fig. 4)	Area of unit circle
4 x triangle AOB =	2.000000000000000...
8 x triangle AOD =	2.828427124746190...
16 x triangle AOF =	3.061467458920718...
32 x triangle AOH =	3.121445152258052...
64 x triangle AO J =	3.136548490545939...
128 x triangle AOL =	3.140331156954753...
256 x triangle AON =	3.141277250932773...
512 x triangle AOP =	3.141513801144301...
. . .	. . .

When we print and analyze the "h" values of each hypotenuse as per computer program step 50 above, we find that a recursive, convergent, infinite sequence of values exists between each term of the hypotenuse.

a	b	h	differential
1	1	1.4142135623...	
			$\sqrt{\sqrt{2}+2}$
0.2928932188...	0.7071067811...	0.7653668647...	
			$\sqrt{\sqrt{\sqrt{2}+2}+2}$
0.0761204674...	0.3826834323...	0.3901806440...	
			$\sqrt{\sqrt{\sqrt{\sqrt{2}+2}+2}+2}$
0.0192147195...	0.1950903220...	0.1960342806...	
			$\sqrt{\sqrt{\sqrt{\sqrt{\sqrt{2}+2}+2}+2}+2}$
0.0048152733...	0.0980171403...	0.0981353486...	
...	...	...	...

Analysis of the recursive increase to the differential shown in the chart above leads to the generation of a new computer program on the basis that every new term of the infinite *P* sequence is twice the previous term divided by the new differential as shown by steps 30 and 40 below. This new computer program provides a printout corresponding to the previous printout for the terms of the infinite *P* sequence as they approach the limit π and, in addition, provides a printout for the terms of the infinite *G* sequence as they approach the limit ϕ.

step	Command
10	P=2
20	$for\ t = 1\ to\ \infty$
30	$A = \frac{2}{\sqrt{D+2}}$
40	$P = P \cdot A$
50	$G = \sqrt{G + A}$
60	$D = \sqrt{D+2}$
70	Print P, Print G
80	next t
90	end

***P* Terms**	***G* Terms**
2.000000000000000...	
2.828427124746190...	1.189207115002721...
3.061467458920718...	1.507182575302380...
3.121445152258052...	1.589582880352798...
3.136548490545939...	1.610720786706718...
3.140331156954753...	1.616145656547426...
3.141277250932773...	1.617543485841641...
3.141513801144301...	1.617905680091623...
...	...
3.14159265358979323...	1.61803398874989

$\cdots$ $\cdots$

π ϕ

The terms of the sequences are shown above in decimal form as approximations of their value. All terms of the two sequences are irrational numbers, but they are considered algebraic irrational numbers because they are the roots of algebraic equations such as $x^2 - 2 = 0$ in which they have rational coefficients. The German mathematician Cantor defined irrational numbers as limits of appropriate sequences of rational numbers. Since these terms are all algebraic irrational numbers they can be constructed by use of an unmarked straightedge and compass. Take a close look at the numerators and denominators of the first three terms of the *P* sequence.

$$\frac{4}{\sqrt{2}},\ \frac{8}{\frac{\sqrt{2}}{\sqrt{\sqrt{2}+2}}},\ \frac{16}{\frac{\frac{\sqrt{2}}{\sqrt{\sqrt{2}+2}}}{\sqrt{\sqrt{\sqrt{2}+2}+2}}}$$

The numerator of each new term is doubled and it is easy to see, by inspection, just how the denominators of the fourth, fifth, sixth, etc. terms will be formed.

We show below that π and ϕ are truly the limits of the infinite *P* and *G* sequences by reviewing the computer design commands of lines 30, 40, 50, and 60 as shown in the table above. Since the limits of the sequences are in the same proportion as the terms, we can review the formulas for the terms to ascertain the limits of the sequences.

Command line 60 shows $D = \sqrt{D+2}$; therefore, $D^2 - D - 2 = 0$; and solving for *D*, at the limit, $D = 2$.

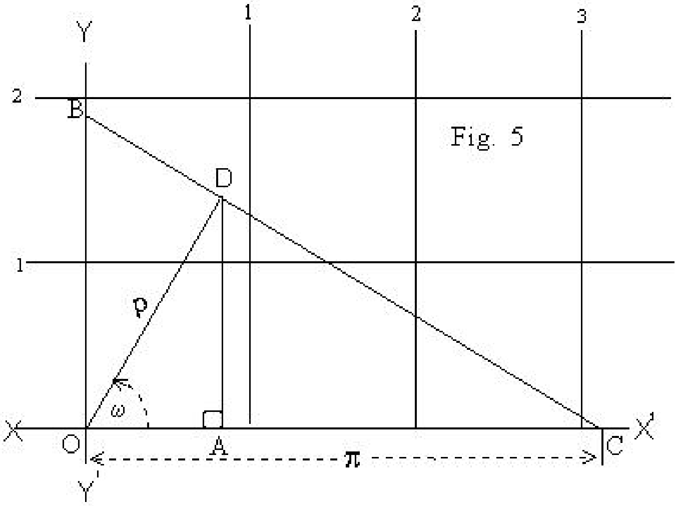
Command line 30 shows $A = \frac{2}{\sqrt{D+2}}$; so, as *D* approaches 2, *A* approaches $\frac{2}{2}$; and, at the limit, $A = 1$.

Command line 40 shows $P = P \cdot A$; so, at the limit, $P = P \cdot 1$; and since the program was designed to compute the area of a unit circle, then, at the limit, $P = \pi$.

Command line 50 shows $G = \sqrt{G + A}$; so, at the limit, $G = \sqrt{G + 1}$; therefore, $G^2 - G - 1 = 0$ and solving for *G* , at the limit, $G = \frac{1+\sqrt{5}}{2}$; which is the number ϕ.

It should be noted that the limit of *A* is $\frac{2}{2}$ where the numerator is a constant 2 and where the denominator approaches the limit 2. Therefore, as the denominator gets larger term by term, the fraction gets smaller and smaller showing that the sequences are converging on the limits of π and ϕ. The sequences are infinite so we cannot reach the limits by calculating endless terms, but we do reach the limit when we show the limit of line segment OA to be a perfect square, a rational number.

Line segment OA shown by Figure 3 above shows the fraction $\frac{bm}{1+m^2}$ to be a simpler form of the fraction $\frac{\phi^2}{\pi}$, such that $\frac{bm}{1+m^2} = \frac{\phi^2}{\pi}$. The equation of a straight line is completely determined if the length of the perpendicular from the origin (0, 0) to the line is known, and if the angle that this perpendicular makes with the x-axis is known. Since the length of the perpendicular p is the limit ϕ and since the limiting ratio for both the sine and cosine of the angle ω are known $\sin\omega = \frac{b}{\phi(1+m^2)}$ and $\cos\omega = \frac{bm}{\phi(1+m^2)}$, we can, by substitution of these values, determine the equation of tangent line BC, and further determine that $\frac{bm}{1+m^2} = \frac{\phi^2}{\pi}$



The normal form of an equation of a straight line is

$$x \cos \omega + y \sin \omega - p = 0.$$

Substituting the known values for the sine, the cosine, and the perpendicular distance in this equation, we determine the equation of the tangent line BC (see Fig. 5) as follows:

Equation of tangent line BC . $\frac{bm}{\phi(1+m^2)}x + \frac{b}{\phi(1+m^2)}y - \phi = 0$

Simplifying . $bmx + by - \phi^2(1+m^2) = 0$

Letting Y = 0, $x = \frac{\phi^2(1+m^2)}{bm}$

But at the limit $x = \pi$ (showing the composite nature of π) $\pi = \frac{\phi^2(1+m^2)}{bm}$

Rearrange by cross multiplication . Q.E.D. $\frac{bm}{1+m^2} = \frac{\phi^2}{\pi}$

When a quadratic equation has its coefficients in such form as to produce a perfect square, the discriminant $\sqrt{b^2 - 4ac} = 0$ and the roots of the equation are calculated by the formula $\frac{-b}{2a}$ where a b and c are coefficients of the equation and $a \neq 0$.

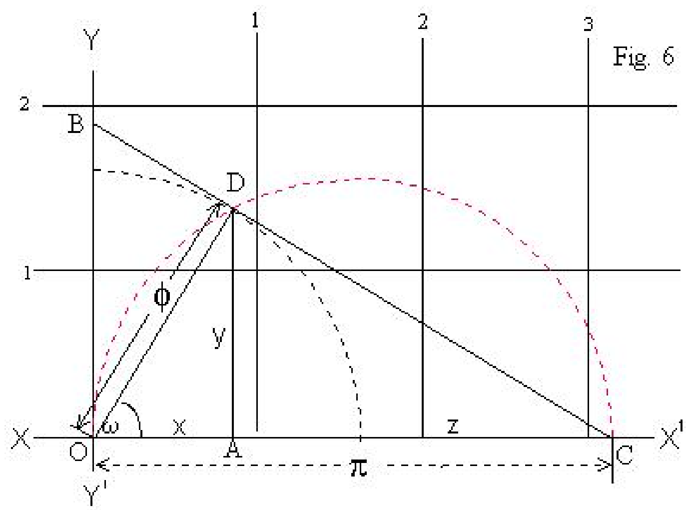
Please note on Figure 3 (see Fig 3. above) that $\overline{OD} = \frac{b}{\sqrt{1+m^2}} = \phi$; so, by substituting $\frac{b^2}{1+m^2}$ for ϕ^2 in the equation of the tangent line $bmx + by - \phi^2(1+m^2) = 0$, we find that the equation of the tangent line reduces to $y = mx + b$. Then, by squaring both sides we obtain $y^2 = m^2x^2 + 2bmx + b^2$, and by substituting this y^2 value of the tangent line for the y^2 value of the circle $y^2 = \phi^2 - x^2$, we derive $x^2 + m^2x^2 + 2bmx + b^2 - \phi^2 = 0$ and again by substituting $\frac{b^2}{1+m^2}$ for ϕ^2, we show the equation $(1 + m^2)x^2 + 2bmx + \frac{b^2m^2}{1+m^2} = 0$ and we note that the coefficients of this equation are in the form of a perfect square as described in the preceding paragraph. Therefore, the limiting equation has dual roots approaching as its limit the perfect square $\frac{bm}{1+m^2}$, and line segment OA is shown to be constructible.

Both π and ϕ^2 are composite numbers each having the same irrational factor $\frac{b}{m(1+m^2)}$ such that $\pi = (1 + m^2)(\frac{b}{m(1+m^2)})$ and $\phi^2 = bm(\frac{b}{m(1+m^2)})$. Both π and $\phi^2$are, and always will be, irrational numbers except when they are put in the form $\frac{\phi^2}{\pi}$or its reciprocal $\frac{\pi}{\phi^2}$, wherein the irrational factors cancel leaving a perfect square $\frac{bm}{1+m^2}$ shown by this mathematical expression $\frac{bm}{(1+m^2)} \cdot \frac{\frac{b}{m(1+m^2)}}{\frac{b}{m(1+m^2)}} = \frac{\frac{b^2m}{m(1+m^2)}}{\frac{b(1+m^2)}{m(1+m^2)}} = \frac{\frac{b^2}{1+m^2}}{\frac{b}{m}} = \frac{\phi^2}{\pi}$ or vice versa, $\frac{\pi}{\phi^2}$. Also, since $\phi = \frac{b}{\sqrt{1+m^2}}$ we may substitute it for ϕ in the expression $\frac{\phi m}{\sqrt{1+m^2}}$ to obtain $\frac{bm}{1+m^2}$; therefore, $\frac{\phi m}{\sqrt{1+m^2}}$ is shown as a simpler form of $\frac{bm}{1+m^2}$, such that $\frac{\phi m}{\sqrt{1+m^2}} = \frac{bm}{1+m^2} = \frac{\phi^2}{\pi}$.

The definition of a composite number is an integer which is the product of two other integers greater than 1. This paper clearly indicates that the fraction $\frac{bm}{1+m^2}$ is a perfect square and every perfect square is a rational fraction with integers for both numerator and denominator and since $\frac{bm}{1+m^2} = \frac{\phi^2}{\pi}$, then, by the rules of mathematics, ϕ^2, π, bm, and $(1 + m^2)$ are all integers when put in this form. We can now show that the integer $\pi$is the product of two other integers $\frac{\phi^2}{1}$ and $\frac{1+m^2}{bm}$ (each factor greater than 1); and since $\phi^2 = \frac{b^2}{1+m^2}$, by substitution, the equation of the composite number may also be written as $\pi = \frac{b^2}{1+m^2} \cdot \frac{1+m^2}{bm}$ or as $\pi = \frac{b}{m}$. In any event the integer π is shown to be the product of two other integers (each factor greater than 1).

When those numbers identified with the Greek letters π and ϕ^2 are put in the form $\frac{\phi^2}{\pi}$, it is extremely difficult, because of the set of our minds, to look upon this fraction as a perfect square, or to accept and understand the fact that the square root of every perfect square is a rational fraction with integers for both the numerator and denominator. It is even more difficult to accept the fact that this fraction $\frac{\phi^2}{\pi}$ is not even in its simplest form. Sometimes rational fractions $(\frac{n}{d})$ are hard to recognize especially when their numerators and denominator are put in the irrational form of $n\sqrt{a}$ or $d\sqrt{a}$ (where a is any natural number except a perfect square). Then, when these numbers are put in fractional form $\frac{n\sqrt{a}}{d\sqrt{a}}$, the irrational factors cancel, the Greek letters disappear, and we are left with the rational fraction $\frac{n}{d}$. The value of a fraction does not change when its numerator and denominator are multiplied by the same quantity; but it may not be easy to tell if it is rational or not just by looking at it; for example, the lowest forms of $\frac{6\sqrt{13}}{15\sqrt{13}}$ and $\frac{\sqrt{51}\cdot\sqrt{17}}{23\sqrt{3}}$ are the rational fractions $\frac{2}{5}$ and $\frac{17}{23}$, respectively.

Since we have shown the transcendental and irrational number π to be a constructible number then, a line segment equal to $\sqrt{\pi}$ is constructible and a square equal to the area of a unit circle is constructible by the method shown by Figure 1 above. A line segment equal $\frac{\pi}{2}$ is also constructible; therefore, a semicircle constructed above the diameter $\overline{OC}$ having a radius of $\frac{\pi}{2}$ will intersect the equation of the circle $x^2 + y^2 = \phi^2$ at the point of tangency $D_n\left(\frac{\phi^2}{\pi}, \sqrt{\frac{\phi^2\pi^2 - \phi^4}{\pi^2}}\right)$ as shown below (Fig. 6).



The limits of the G and P sequences are in the same proportion one to another as their terms. Therefore, π is to ϕ as ϕ is to x and ϕ is shown to be the mean proportional between π and line segment OA.

When we substitute the y^2 value of the circle, $x^2 + y^2 = \phi^2$, for the y^2 value of the limiting equation of the semicircle $(x - \frac{\pi}{2})^2 + y^2 = (\frac{\pi}{2})^2$, we obtain the equation $x^2 - \pi x + \frac{\pi^2}{4} + \phi^2 - x^2 = \frac{\pi^2}{4}$; and solving for x is an easy task since most of the terms cancel and we are left with $x = \frac{\phi^2}{\pi}$. Next, by substituting this value of x, and solving for y, in the equation $y^2 = \phi^2 - x^2$, we obtain $y = \sqrt{\frac{\pi^2\phi^2 - \phi^4}{\pi^2}}$, confirming the point of tangency as $D_n\left(\frac{\phi^2}{\pi}, \sqrt{\frac{\phi^2\pi^2 - \phi^4}{\pi^2}}\right)$.

This point of tangency, in x and y, matches the point of tangency $D_n(\frac{bm}{1+m^2}, \frac{b}{1+m^2})$ in b and m and since the point of tangency satisfies the equation of the semicircle, we know that the limits of the line segments previously established for the π/ϕ structure are confirmed to be in a classical mean proportion.

Top

THE MAGNIFICENT

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PART 3 (THE PROOF)

THIS COMPOSITE NUMBER FOR PI SATISFIES EULER'S EQUATION.

- **EULER'S PROOF IS NOT ONLY CONFIRMED, IT IS CLARIFIED.**
- **TRUTH EXISTS WHETHER WE FIND IT OR NOT.**

The composite number $\phi^2 \cdot \left[\frac{1+m^2}{bm}\right]$ when substituted for π is shown to satisfy and clarify Euler's famous equation $e^{\pi i} = -1$. Refer to Fig 3. and note for right triangle OAD that $\overline{OA}^2 + \overline{AD}^2 = \overline{OD}^2$; therefore, $\phi^2 = \left(\frac{bm}{1+m^2}\right)^2 + \left(\frac{b}{1+m^2}\right)^2$. This equality allows us to substitute the expression $\left(\frac{bm}{1+m^2}\right)^2 + \left(\frac{b}{1+m^2}\right)^2$ for ϕ^2 in the composite number $\phi^2 \cdot \left[\frac{1+m^2}{bm}\right]$ to obtain $\pi = \left[\left[\frac{bm}{1+m^2}\right]^2 + \left[\frac{b}{1+m^2}\right]^2\right] \cdot \left[\frac{1+m^2}{bm}\right]$; so simplify and multiply through by $\left(\frac{1+m^2}{bm}\right)$ to obtain

$$\pi = \left[\left(\frac{bm}{1+m^2}\right) \cdot \left(\frac{bm}{1+m^2}\right) \cdot \left(\frac{1+m^2}{bm}\right)\right] + \left[\left(\frac{b}{1+m^2}\right) \cdot \left(\frac{b}{1+m^2}\right) \cdot \left(\frac{1+m^2}{bm}\right)\right],$$

then cancel like terms to obtain

$\pi = \left[\left(\frac{bm}{1+m^2}\right)\cdot\left(\frac{1}{1}\right)\cdot\left(\frac{1}{1}\right)\right] + \left[\left(\frac{b}{1+m^2}\right)\cdot\left(\frac{1}{1}\right)\cdot\left(\frac{1}{m}\right)\right]$**, and multiply by i to obtain** $\pi i = \left[\left(\frac{bm}{1+m^2}\right)\cdot\left(\frac{1}{1}\right)\cdot\left(\frac{1}{1}\right)\right]i + \left[\left(\frac{b}{1+m^2}\right)\cdot\left(\frac{1}{1}\right)\cdot\left(\frac{1}{m}\right)\right]i$

Since $\frac{bm}{1+m^2}$ **is a perfect square we may rewrite the previous expression as**

$\pi i = \left[\left(\frac{\sqrt{bm}}{\sqrt{1+m^2}}\right)\cdot\left(\frac{\sqrt{bm}}{\sqrt{1+m^2}}\right)\cdot\left(\frac{1}{1}\right)\cdot\left(\frac{1}{1}\right)\right]i + \left[\left(\frac{b}{1+m^2}\right)\cdot\left(\frac{1}{1}\right)\cdot\left(\frac{1}{m}\right)\right]i$**, or as**

$\pi i = \left[\left(\frac{\sqrt{bm}}{\sqrt{1+m^2}}\right)i\cdot\left(\frac{\sqrt{bm}}{\sqrt{1+m^2}}\right)i\cdot\left(\frac{1}{1}\right)i\cdot\left(\frac{1}{1}\right)i\right] + \left[\left(\frac{b}{1+m^2}\right)i\cdot\left(\frac{1}{1}\right)i\cdot\left(\frac{1}{m}\right)i\right]$**, and then,**

$\pi i = \left[\left(\frac{bm}{1+m^2}\right)i^2\cdot\left(\frac{1}{1}\right)i^2\right] + \left[\left(\frac{b}{1+m^2}\right)i\cdot\left(\frac{1}{1}\right)i\cdot\left(\frac{1}{m}\right)i\right]$**, which can be shown as**

$\pi i = \left[-\left(\frac{bm}{1+m^2}\right)\cdot\left[\frac{1}{1}\right]i^2\right] + \left[\left(\frac{b}{1+m^2}\right)i\cdot\left(\frac{1}{m}\right)i^2\right]$**. Next, using this equation as an exponent for e,**

$e^{\pi i} = e^{\left[-\left(\frac{bm}{1+m^2}\right)^{i^2}\right]+\left[\left(\frac{b}{1+m^2}\right)^{i}\left(\frac{1}{m}\right)^{i^2}\right]}$**, and since** $\left(\frac{1}{m}\right)^{i^2} = \left(\frac{1}{m}\right)^{-1} = \frac{m}{1}$**, this may be written as**

$e^{\pi i} = e^{-\left(\frac{bm}{1+m^2}\right)^{i^2}+\left(\frac{bm}{1+m^2}\right)^{i}}$**, and since** $\frac{bm}{1+m^2}$ **is a perfect square, we may rewrite the equation as**

$e^{\pi i} = e^{-\left(\frac{bm}{1+m^2}\right)^{i^2}+\left(\frac{bm}{1+m^2}\right)^{i^2}}$ **. Finally the equation may be written as**

$e^{\pi i} = \left(e^0\right)^{i^2}$ **and this can be put in the form of** $1\cdot\sqrt{-1}\cdot\sqrt{-1} = -1$ **Q.E.D.**[14]

The search is over; the irrational number ϕ is shown to be the mean proportional between the irrational and transcendental number π and the perfect square $\frac{bm}{1+m^2}$, such that $\frac{\pi}{\phi} = \frac{\phi}{\frac{bm}{1+m^2}}$ or $\pi = \frac{\phi^2(1+m^2)}{bm}$, and π is shown to be both a composite number and a constructible number.

The nature of the numbers π and ϕ has not suddenly changed; the relationship between these two numbers has always been true. Man has always been seeking truth and truth exists whether we find it or not. Most important for man is the truth of the Word of God:

(Rom 3:23 KJV) For all have sinned, and come short of the glory of God;

(John 3:16 KJV) For God loved the world, that he gave his only begotten Son, that whosoever believeth in him should not perish, but have everlasting life.

(John 8:32 KJV) And ye shall know the truth, and the truth shall make you free.

(John 14:6 KJV) Jesus saith unto him, I am the way, the truth, and the life: no man cometh unto the Father, but by me.

(John 16:13 KJV) Howbeit when he, the Spirit of truth, is come, he will guide you into all truth: for he shall not speak of himself; but whatsoever he shall hear, that shall he speak: and he will show you things to come.

(Rev 3:20 KJV) Behold, I stand at the door, and knock: if any man hear my voice, and open the door, I will come in to him, and will sup with him, and he with me.

| Introduction and Table of Contents | | Part 1 - The Narrative | | Part 2 - The Math | | Part 3 - The Proof | | Glossary | | Footnotes |

Top

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THE GLOSSARY

Glossary of Terms:

Algebraic irrational number — An irrational number that can be expressed as a root of an algebraic equation of the 2nd degree, for example $x^2 - 2 = 0$.

Algebraic number — A real number that is the root of an equation in the form

$$c_n x^n + c_{n-1} x^{n-1} + c_{n-2} x^{n-2} + \cdots + c_2 x^2 + c_1 x^1 + c_0 = 0.$$

Complex number — The addition of a real number and a pure imaginary number in the form of a + bi (a complex number) where a and b are real numbers and $i^2 = -1$. When we put b = 0 and $a \neq 0$, the complex number is a real number, and when we put a = 0 and $b \neq 0$ then the complex number is a pure imaginary number.

Composite number — A real number that can be factored, an integer which is the product of two other integers greater than 1.

Constructible number — means constructible by the Euclidean method of straightedge and compass. All algebraic numbers (including algebraic irrationals) of the 1st or 2nd degree, or those having a power of two, e.g., x^2, x^4, x^8, x^{16}, x^{32}, x^{64}, x^{128}, ..., etc., if given a line segment of unit length, are numbers that can be constructed by the classical Greek geometric method of straightedge and compass.

Convergent Sequence — A sequence with a finite limit.

Finite Sequence — A set of limited numbers (terms) such that each successive number is determined according to some definite rule or law.

Imaginary number — A number that can be expressed as a root of an algebraic equation of the 2nd degree as $x^2 + 1 = 0$, but since no real number (rational or irrational) can be squared and then added to1 to obtain zero, the mathematicians invented a new symbol, or unit, known as i, where $i = \sqrt{-1}$. If any real number is multiplied by i, a pure imaginary number results, for example $5i = 5 \cdot \sqrt{-1}$. (Also see the definition of a complex number above.)

Infinite Sequence — A set of unlimited numbers (terms) such that each successive number is determined according to some definite rule or law.

Integer — Any natural number plus zero and the negative numbers.

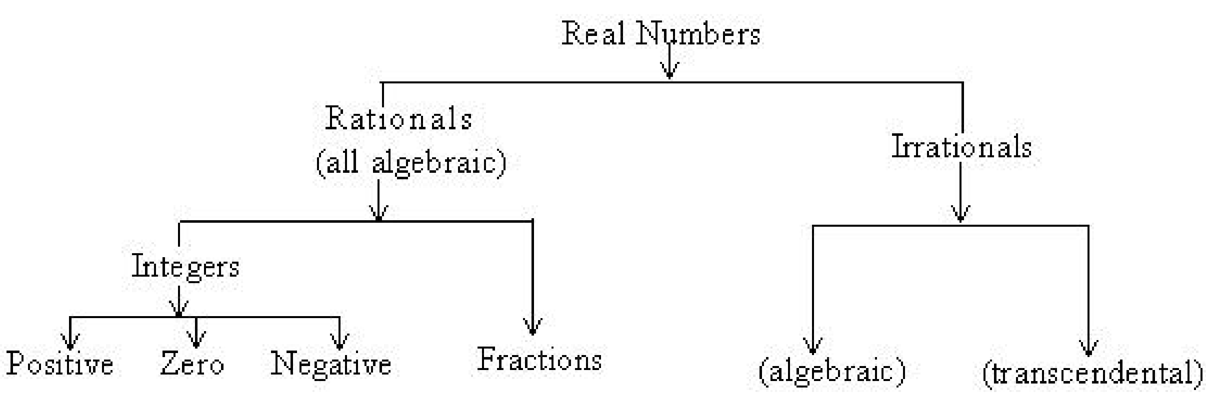
Irrational number — A real number that cannot be expressed as the ratio of two integers.

Limit of a sequence — A finite numerical value on which the terms of a sequence are converging. For example, if the radius of a unit circle starts at $\frac{1}{2}$ and increases term by term without limit, as shown by the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \cdots$, it is seen that as the additional terms become larger the radius of the circle approaches closer and closer to the limit 1.

Natural number — Any positive whole number, the smallest being one. There is no largest.

Rational number — A number that can be put in the form of $\frac{n}{d}$ (*n* and *d* are integers and $d \neq 0$).

Real numbers — The collection of all numbers associated with all of the points on a line. This collection includes both rational and irrational numbers, and is a subclass of the complex number system. It is best summarized by the following diagram of the entire real number system.



Recursive Sequence — A sequence having one or more of its terms specified, and then each successive term defined by means of the preceding terms.

Series — A sequence having each term connected by a plus or minus sign.

Transcendental number — An irrational number that cannot be expressed as the root of an algebraic equation having rational coefficients.

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THE FOOTNOTES

Footnotes:

[1] Approximately 3000 – 4000 years ago.

[2] Golovina L.I. and Yaglom I.M.: Induction in geometry. In Topics in Mathematics. Boston, D.C. Heath and Company, 1963, p. 2.

[3] Adler I.: The New Mathematics. New York, New American Library, Inc., 1959, p. 35.

[4] Adler, p.108.

The reader is advised that the words nesting points are coined from Mr. Adler's comments on page 108 quoted as follows: "...: an infinite decimal represents a nest of intervals on the line, and for every such nest, there is one and only one point that lies inside every interval of the nest...."

[5] Ayre H.G.: Basic Mathematical Analysis, New York, McGraw-Hill, 1950. p. 17.

[6] The new Encyclopedia Britannica-- 15th edition, 1991. Copyright by Encyclopedia Britannica, Inc., 1991

[7] Ayre, p. 11.

[8] Ayre, p. 13.

[9] Ayre, p. 11.

[10] Adler, p.178.

[11] Niven I.: Numbers: Rational and Irrational, Washington D.C., The Mathematical Association of America, 1961, p. 71.

[12] Niven , p.76.

[13] Eves H.: An Introduction to the History of Mathematics. New York, Holt, Rinehart and Winston, Inc., 1964, p. 82.

[14] Consider the fact that no one has been here before. There are no prior rules for this situation. The mathematics must be accepted since it logically confirms Euler's formula $e^{\pi i} = -1$. When the composite number $\phi^2 \cdot \left\lceil \frac{1+\pi^2}{3\pi} \right\rceil$ is substituted for π in Euler's equation $e^{\pi i} = -1$, Euler's proof is not only confirmed, it is clarified. Things equal to the same thing are equal to each other. Please also note that the number e plays no apparent role in the relationship of these numbers. It should be obvious to all readers that any number raised to the power of πi will equal – 1.

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